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# On vector bundles and complete intersections of finite order

By

Takeo OHSAWA

## 1. Introduction.

The present paper will be an introduction to the theory of M.Cornalba and P.Griffiths [ 1] on holomorphic vector bundles of finite order, with a proof of the following result.

Theorem  Any smooth algebraic curve in $\mathbb{C}^n$ is the intersection of two surfaces of finite order meeting transversely.

This result is an affirmative answer to a conjecture of Cornalba and Griffiths (cf. §26 in [ 1 ]). For a detailed proof with a relevant generalization, see the paper of O.Forster and the author [2 ].

## 2. Functions of finite order on affine algebraic manifolds.

Let $X$ be an affine algebraic manifold of dimension $n$ embedded in a complex number space $\mathbb{C}^N$. A holomorphic function $f$ on $X$ is said to be of finite order if there exist a holomorphic function $F$ on $\mathbb{C}^N$ and a polynomial $P$ of real one variable such that $F|X = f$ and $|F(z)| < e^{P(\|z\|)}$, where $z$ denotes the coordinate and $\|z\|$ denotes the norm of $z$.

More intrinsic definition of a function of finite order is the following.

Definition.  A holomorphic function $f$ on $X$ is of finite order if there exists a polynomial $P$ of real one variable satisfying $f(x) < e^{P(\|x\|)}$ for any $x \in X$.

Equivalence of these definitions can be seen by using a cohomology vanishing theorem with growth conditions (cf. L.Hörmander's book "An Introduction to Complex Analysis in Several Variables").

Another method of defining functions of finite order is the following. By Hironaka's theorem there exists a projective smooth compactification $\overline{X}$ of $X$ such that $D = \overline{X} \setminus X$ is the union of smooth divisors with normal crossings: for every point $p$ $D$ one can find a neighbourhood $U \ni p$ such that $(U, U \setminus X)$ is biholomorphic to $(\Delta^n, \Delta^n \setminus \{z_1 \cdot \dots \cdot z_k = 0\})$, where $\Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n;\ |z_i| < 1 \ \ \forall i\}$. A function $f$ on $X$ is of finite order if it satisfies, for every $p \in D$, an estimate of the type $|f(z)| < \exp(\sum_{i=1}^{k} |z_i|^{-r})$ near $p$, for some $r \in N$. We shall denote by $R_X$ the ring of holomorphic functions of finite order on $X$.

<u>Proposition 1.</u> Let $X$ be an affine algebraic manifold and let $U_1, \dots, U_m$ be a Zariski open affine covering of $X$. Then a holomorphic function $f$ on $X$ belongs to $R_X$ if and only if $f|U_i \in R_{U_i}$ for all $i$.

A classical theorem of Weierstrass asserts that every nowhere vanishing holomorphic function of finite order on $\mathbb{C}$ is the exponential of some polynomial. As its immediate consequence, we get that an element of $R_{\mathbb{C}}$ is invertible if and only if it has no zeroes. The following generalization is immediate.

<u>Proposition 2.</u> Let $X$ be an affine algebraic manifold and $f$ a function of finite order on $X$ without zeroes. Then $1/f$ is also a function of finite order.

Let $\theta^r$ denote the trivial bundle of rank $r$. For any subbundle $E \subset \theta^r$ a holomorphic section $s \in \Gamma(X,E)$ is represented by an $r$-tuple $(s_1, \dots, s_r)$ of holomorphic functions. The section $s$ is said to be of finite order, if all functions $s_j$ are of finite order. If $E$ is an algebraic vector bundle, then there exist $r$ and an algebraic embedding $E \subset \theta^r$. It is easy to verify that the set of finite order sections is independent of the (algebraic) embeddings, and we denote it by $\Gamma_{f.o.}(X,E)$. Proposition 2 is generalized to the following.

<u>Proposition 3.</u> Let $L \subset \theta^r$ be an algebraic subbundle of rank one over an affine algebraic manifold $X$. Let $s = (s_1, \dots, s_r) \in \Gamma_{f.o.}(X,L)$ be a section such that $s_1, \dots, s_r$ have no common zeroes. Then there exist $f_1, \dots, f_r \in R_X$ such that $f_1 s_1 + \dots + f_r s_r = 1$.

Cornalba and Griffiths have shown that topologically trivial algebraic line bundles on affine algebraic manifolds are trivial in the finite order category. In the following section we shall state it more accurately after explaining the notion of holomorphic vector bundles of finite order.

3. Holomorphic vector bundles with growth conditions.

Given a holomorphic vector bundle $E$ over an affine algebraic manifold $X$ embedded in $\mathbb{C}^N$, we say that $E$ has finite order if there exists a fiber metric for this bundle whose curvature form $\Theta$ satisfies the inequality $|\Theta(x)| < (2+\|x\|)^r$ for some $r \in \mathbb{R}$. The following is immediate from the definition.

<u>Proposition 4.</u> Given two finite order vector bundles $E$, $E'$ over $X$, $\mathrm{Hom}(E,E')$, $E \oplus E'$, $E \otimes E'$ and $E^*$ are of finite order.

It is easy to show that every algebraic vector bundle is of finite order.

To state Cornalba-Griffiths' theorem in a precise form, let $E \longrightarrow X$ be a holomorphic vector bundle of finite order. Using the given fiber metric of $E$, we define holomorphic sections of finite order (of exponential growth in the $L^2$-sense) by

$$L_{f.o.}(X,E) := \left\{ \beta \in \Gamma(X,E);\ \int_X |\beta|^2 e^{-\|x\|^k} dv_X < \infty \ \text{ for some } k \right\},$$

where $dv_X$ denotes the volume element of $X$ with respect to the ambient euclidean metric. By Proposition 4, the notion of finite order isomorphism between two finite order bundles is defined. Thus the set of equivalence classes $\mathrm{Vect}^r_{f.o.}(X)$ of finite order vector bundles of rank $r$ over $X$ has a meaning. In particular, for $r=1$ we define the finite order Picard variety by $\mathrm{Pic}_{f.o.}(X) = \mathrm{Vect}^1_{f.o.}(X)$.

<u>Theorem 1.</u> Let $X$ be an affine algebraic manifold. Then the Chern class map $c_1 : \mathrm{Pic}_{f.o.}(X) \longrightarrow H^2(X,Z)$ is an isomorphism.

For the proof, the reader is referred to §14 or §§19-20 of [ 1 ].

Remark Our way of defining the set $L_{f.o.}(X,E)$ is slightly different from that in [ 1 ]. But the two notions of finite order sections coincide, because the volume form $dv_X$ in the definition may be replaced by any other volume form $dv'$ satisfying $(2+\|x\|)^{-r}dv_X < dv' < (2+\|x\|)^r dv_X$ for some $r > 0$.

The following is a consequence of Theorem 1.

Theorem 2. Let $L$ be an algebraic line bundle over an affine algebraic manifold $X$. Suppose $L$ is topologically trivial. Then there exists a finite order section $s \in \Gamma_{f.o.}(X,L)$ without zeroes on $X$.

Proof. Let $\theta$ be the trivial line bundle over $X$ equipped with the trivial metric. Since $L^*$ is also topologically trivial, by Theorem 1 one has a finite order section $s \in L_{f.o.}(X,(L^*)^*\otimes\theta) = L_{f.o.}(X,L)$ without zeroes. Here one may take as the fiber metric of $L$ the restriction of the trivial metric with respect to some algebraic embedding $L \subset \theta^r$. It is clear that $L_{f.o.}(X,L) = \Gamma_{f.o.}(X,L)$ for such a metric. Thus we have the assertion.

4. An extension and division theorem.

We are going to prove the following.

Theorem 3. Let $X$ be an algebraic submanifold of $\mathbb{C}^N$ of pure dimension $n$ such that the projection $p : X \longrightarrow \mathbb{C}^n$ to the first $n$ coordinates is proper. Let $\{g_1,\ldots,g_r\} \subset \mathbb{C}[z_1,\ldots,z_N]$ be a set of generators of the ideal of polynomials vanishing on $X$. Then the natural restriction map

$$\pi : R_{\mathbb{C}^n}[z_{n+1},\ldots,z_N] \longrightarrow R_X$$

is surjective and its kernel is generated by $g_1,\ldots,g_r$.

Our proof will be done by the $L^2$ method. First we need to interpret the problem into the $L^2$ language.

Let $X$ and $p : X \longrightarrow \mathbb{C}^n$ be as above. We put $\varphi(z) = \log(1 + \|z\|^2)$. Then, $\partial\bar\partial\varphi$ defines the Fubini-Study metric of $\mathbb{P}^N$, whose volume element we shall denote by $dv_{\varphi,X}$. The volume element on $X$ for the metric $\partial\bar\partial\varphi | X$ will be denoted by $dv_{\varphi,X}$. Comparison of $dv_{\varphi,X}$ with the euclidean volume element $dv_X$ is given by: $dv_{\varphi,X} < dv_X < \text{const}\ e^{(n+1)\varphi}\, dv_{\varphi,X}$.

Definition. A $C^\infty$ plurisubharmonic function $\eta : X \longrightarrow \mathbb{R}$ is called a controlling function if there exists a $\nu \in N$ such that $|\eta(x) - \eta(y)| < 1$ whenever $\|x - y\| < (2+\|x\|)^{-\nu}$.

Note that for any controlling function $\eta$ on X and for any $\nu \in N$, $\eta + \nu\varphi$ is again a controlling function. We put, for a controlling function $\eta$,

$$R_X(\eta) = \left\{ f \in R_X \ ; \ \sup |f(x)|^2 \exp(-\eta(x) - \nu\varphi(x)) < \infty \quad \text{for some} \quad \nu \in N \right\}.$$

For the special controlling functions $\|z\|^{2r}$, we have

$$R_X = \bigcup_{r \in N} R_X(\|z\|^{2r}).$$

For the controlling functions $\psi_r(z) = (|z_1|^2 + \cdots + |z_n|^2)^r$ we have

$$R_{\mathbb{C}^n}[z_{n+1}, \ldots, z_N] = \bigcup_{r \in N} R_{\mathbb{C}^N}(\psi_r).$$

Since the projection p is proper, we have also

$$R_X = \bigcup_{r \in N} R_X(\psi_r).$$

Allpying a standard $L^2$ method for extending holomorphic functions from a complex submanifold we obtain the following (cf.[ 2 ] or [ 6 ]).

Proposition 5. Let M be an affine algebraic manifold, and let $X \subset M$ be an algebraic submanifold. Then, for any controlling function $\eta$ on M, there exists a $\nu \in N$ such that each $f \in \Gamma_\eta(X, \mathcal{O}_X)$ has an extension F in $\Gamma_{\eta+\nu\varphi}(M, \mathcal{O}_M)$.

Corollary 6. For any controlling function $\eta$ on M, the restriction map $\pi : R_M(\eta) \longrightarrow R_X(\eta)$ is surjective.

To determine the kernel of $\pi : R_M(\eta) \longrightarrow R_X(\eta)$, we need Skoda's division theorem for holomorphic vector bundles over weakly 1-complete manifolds (cf. [ 4 ] Théorème 4). We describe it for the trivial bundle over Stein mqnifolds since we only need that special case.

Theorem 4. Let $M$ be a Stein manifold of dimension $m$ and let $f_1,\dots,f_q$ be holomorphic functions on $M$ with $q < m+1$. Let $\Phi$ be a $C^\infty$ strictly plurisubharmonic function on $M$. Then, given a plurisubharmonic function $\psi$ on $M$ and a holomorphic $m$-form $u$ such that

$$\left| \int_M e^{-\psi-\Phi} \left( \sum_{i=1}^q |f_i|^2 \right)^{-q} u \wedge \bar{u} \right| < \infty ,$$

there exists a $q$-tuple of holomorphic $m$-forms $(h_1,\dots,h_q)$ such that

$$u = \sum_{j=1}^q f_j h_j$$

and

$$\left| \int_M e^{-\psi-\Phi} \left( \sum_{i=1}^q |f_i|^2 \right)^{-q+1} h_j \wedge \bar{h}_j \right| < \left| \int_M e^{-\psi-\Phi} \left( \sum_{i=1}^q |f_i|^2 \right)^{-q} u \wedge \bar{u} \right| .$$

As a direct consequence of Theorem 4 we obtain the following.

Proposition 7. Let $X \subset M \subset \mathbb{C}^n$ be algebraic submanifolds and $\{ g_1,\dots,g_r \}$ be a set of generators of the ideal of $X$ in the affine algebraic coordinate ring of $M$. Then, for any controlling function $\eta$ on $M$, the kernel of the restriction map $\pi : R_M(\eta) \longrightarrow R_X(\eta)$ is $\sum_{i=1}^r R_M(\eta) g_i$.

Proof. Let $f \in \operatorname{Ker} \pi$ and let $q$ be the codimension of $X$ in $M$. If $\operatorname{rank}(dg_{i_1},\dots,dg_{i_q})(x) = q$, then $|f|^2 \left( \sum_{k=1}^q |g_{i_k}|^2 \right)^{-q}$ is integrable on a neighbourhood of $x$. Since $g_i$ are algebraic we can choose an algebraic $m$-form $\omega$ on $M$ such that $|f|^2 \left( \sum_{k=1}^q |g_{i_k}|^2 \right)^{-q} \omega \wedge \bar{\omega}$ is locally integrable. Hence we can apply Theorem 6 for $u = f\omega$ and $(f_1,,,f_q) = (g_{i_1},\dots,g_{i_q})$, and get a $q$-tuple $(h_{i_1},\dots,h_{i_q})$ of holomorphic $m$-forms on $M$ such that

$$(\dagger) \qquad f\omega = \sum_{k=1}^q h_{i_k} g_{i_k}$$

and

$$\left| \int_M e^{-\eta-\nu\varphi} h_{i_k} \wedge \overline{h_{i_k}} \right| < \infty \qquad \text{for all } k, \text{ for some } \nu \in \mathbb{N}.$$

Since $x$ is arbitrary, one can find a finite number of algebraic $m$-forms $\omega_1, \ldots, \omega_s$ on $M$ without common zeroes, such that $f\omega_j$ is a linear combination of $g_i$ as in $(\dagger)$, which shows that $f$ must lie in the ideal generated by $g_1, \ldots, g_r$.

Combining Proposition 7 with Corollary , we obtain Theorem 3.

4. Curves in $\mathbb{C}^n$.

In this section we shall prove Cornalba-Griffiths' conjecture in the affirmative.

Let $X \subset \mathbb{C}^n$ be a smooth affine algebraic curve. By a linear coordinate change of $\mathbb{C}^n$, we choose a coordinate such that the projection $p : X \longrightarrow \mathbb{C}$ to the first coordinate is proper. We shall show the following.

<u>Theorem</u> The ideal of $X$ is generated by $n-1$ elements $F_1, \ldots, F_{n-1} \in R_{\mathbb{C}}[z_2, \ldots, z_n]$.

Proof. This is trivial for $n < 3$. Suppose $n \geq 3$. Then there exist two linear functions $l_1$ and $l_2$ on $\mathbb{C}^n$ such that $(p, l_1, l_2) : X \longrightarrow \mathbb{C}^3$ is an embedding. So we may assume that $p$ factors as

$$\begin{tikzcd} X \arrow[r, hook] \arrow[drr, "p"'] & \mathbb{C}^n \arrow[r, "p_2"] & \mathbb{C}^3 \arrow[d, "p_1"] \\ & & \mathbb{C} \end{tikzcd}$$

where $p_1$, $p_2$ are projections and $p_2$ maps $X$ isomorphically onto the smooth algebraic curve $Y := p_2(X)$. Since $X \subset Y \times \mathbb{C}^{n-3}$ is a graph over $Y$, there exist polynomials $G_1, \ldots, G_{n-3} \in \mathbb{C}[z_1, \ldots, z_n]$ such that the restrictions $G_j|Y \times \mathbb{C}^{n-3}$ generate the ideal of $X$ in $Y \times \mathbb{C}^{n-3}$. Therefore if we could show that there exist functions $F_1, F_2 \in R_1[z_2, z_3]$ which generate the ideal of $Y \subset \mathbb{C}^3$ the proof is complete, because then $F_1, F_2, G_1, \ldots, G_{n-3}$ generate the ideal of $X$ in $\mathbb{C}^n$. Thus we may assume $n=3$.

Let $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{C}^3}$ be the ideal sheaf of $X$. Then $\mathcal{I}_X$ admits a finite set of polynomial generators $g_1, \ldots, g_{N+1} \in \mathbb{C}[z_1, z_2, z_3]$. Let $g : \mathcal{O}_{\mathbb{C}^3}^{N+1} \longrightarrow X$ be the epimorphism defined by $(g_1, \ldots, g_{N+1})$. Since $X$ is of codimension 2, the kernel of $g$ is locally free, hence globally free (algebraically) by the theorem of Quillen-Suslin [3,5]. Hence we have an exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}^N_{\mathbb{C}^3} \longrightarrow \mathcal{O}^{N+1}_{\mathbb{C}^3} \xrightarrow{g} \mathcal{I}_X \longrightarrow 0,$$

where

$$P = \begin{pmatrix} P_{1,1} & \cdots\cdots & P_{1,N} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ P_{N+1,1} & \cdots\cdots & P_{N+1,N} \end{pmatrix}$$

is an $(N+1)\times N$ matrix with coefficients in $\mathbb{C}[z_1,z_2,z_3]$. Restricting the sequence $(*)$ to $X$, we get an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^N_X \xrightarrow{\overline{P}} \mathcal{O}^{N+1}_X \longrightarrow \nu_X \longrightarrow 0,$$

where $\mathcal{K}$ is an algebraic subbundle of $\mathcal{O}^N_X$ and $\overline{P} = P|X$.

Since every vector bundle over a noncompact Riemann surface is topologically trivial, by Theorems 1 and 2 there exists a finite order section

$$(\overline{\varphi}_1,\ldots,\overline{\varphi}_N) \in \Gamma_{f.o.}(X,\mathcal{K}) \subset \Gamma_{f.o.}(X,\mathcal{O}^N_X)$$

without zeroes. By Proposition 3 there exists an N-tuple $(\overline{f}_1,\ldots,\overline{f}_n) \in \Gamma_{f.o.}(X,\mathcal{O}^N_X)$ with $\overline{\varphi}_1\overline{f}_1 + \ldots + \overline{\varphi}_N\overline{f}_N = 1$. Let $f_1,\ldots,f_N \in R_{\mathbb{C}}[z_2,z_3]$ be functions with $f_j|X = \overline{f}_j$ (cf. Theorem 3). We put $f = (f_1,\ldots,f_N)$ and

$$Q = \begin{pmatrix} P \\ f \end{pmatrix}.$$

Then $Q$ is an $(N+2)\times N$ matrix with coefficients in $A := R_{\mathbb{C}}[z_2,z_3]$.

Lemma 8. The cokernel of the map $Q : A^N \longrightarrow A^{N+2}$ is a free A-module of rank 2.

For the moment let us admit Lemma 8 and show how it implies Theorem.

Since 'Coker $Q$ is free' $\Longleftrightarrow$ $\exists$ an A-valued matrix $h$ s.t. $\det(Q,h)=1$, we have the following diagram:

$$
\begin{tikzcd}
 & \mathcal{O}^2_{\mathbb{C}^3} \arrow[d, "h"] & & \\
\mathcal{O}^N_{\mathbb{C}^3} \arrow[r, "\binom{P}{f}"] \arrow[d, equal] & \mathcal{O}^{N+1}_{\mathbb{C}^3} \oplus \mathcal{O}_{\mathbb{C}^3} \arrow[d, "pr"] & & \\
\mathcal{O}^N_{\mathbb{C}^3} \arrow[r, "P"] & \mathcal{O}^{N+1}_{\mathbb{C}^3} \arrow[r, "g"] & \mathcal{J}_X \arrow[r] & 0
\end{tikzcd}
$$

where $pr$ is the projection to the first $N+1$ components. Thus we obtain an epimorphism $g \circ pr \circ h : \mathcal{O}^2_{\mathbb{C}^3} \longrightarrow \mathcal{J}_X$, which proves our assertion.

For the proof of Lemma 8 we first show the following.

<u>Lemma 9.</u> $\operatorname{Coker} Q$ is a projective $A$-module of rank 2.

Proof. It suffices to show that $Q$ is of rank $N$ at every point of $\operatorname{Spec}(A)$. This follows immediately from Theorem 3.

In order to deduce Lemma 9 from Lemma 8, we apply the following theorem of Quillen-Suslin [3 ,5 ].

<u>Theorem 5.</u> Let $B$ be a commutative ring and $A:=B[T]$; the polynomial ring in one indeterminate over $B$. Let $E$ be a finitely generated projective $A$-module. If there exists a monic polynomial $g \in A$ such that the localized module $E_g$ is free over $A_g$, then $E$ is free over $A$.

We apply Theorem 5 to $B = R_{\mathbb{C}}[z_2]$, $A=R_{\mathbb{C}}[z_2,z_3]$ and $E = \operatorname{Coker} Q$. Let $g$ be a polynomial, monic with respect to $z_3$, which vanishes on $X$. Then there exist $\alpha_1,\dots,\alpha_{N+1} \in \mathbb{C}[z_1,z_2,z_3]_g$ such that

$$
\det \left( P \;\middle|\; \begin{matrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_{N+1} \end{matrix} \right) = 1 .
$$

Therefore we have also

$$
\det \left( \begin{array}{c|c|c} P & \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_{N+1} \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline f_1 \cdots f_N & 0 & 1 \end{array} \right) = 1.
$$

The latter matrix has coefficients in $A_g$. This implies that $E_g$ is a free $A_g$-module of rank 2. By Theorem 5, E is free over A.

Q.E.D.

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